Eur Phys J C **41**, s01, 1–4 (2005)
Digital Object Identifier (DOI) 10.1140/epjcd/s2005-01-001-9

EPJ C direct
electronic only

Original Article / Theoretical Physics

Negative phase velocity of electromagnetic waves and the cosmological constant

T.G. Mackay[1], S. Setiawan[1], A. Lakhtakia[2,a]

[1] School of Mathematics, University of Edinburgh, Edinburgh EH9 3JZ, UK, e-mail: T.Mackay@ed.ac.uk
[2] Department of Engineering Science and Mechanics, Pennsylvania State University, University Park, PA 16802–6812, USA

Received: 23 March 2005 / Revised version: 9 April 2005 /
Published online: 22 April 2005 –

Abstract. Examining the propagation of electromagnetic plane waves with the wavevector directed in opposition to the time–averaged Poynting vector in cosmological spacetime with piecewise uniform metric, we show that such negative–phase–velocity (NPV) propagation is possible in certain de Sitter spacetimes but not in anti–de Sitter spacetimes. This difference suggests the possibility of an optical/electromagnetic experiment to discern the cosmological constant of a four–dimensional universe stemming from a five–dimensional brane universe.

PACS. 41.20.Jb Electromagnetic wave propagation; radiowave propagation – 04.40.Nr Einstein-Maxwell spacetimes, spacetimes with fluids, radiation or classical fields – 98.80.-k Cosmology

1 Introduction

Literature on the phenomenon of negative refraction by certain linear, homogeneous, isotropic, dielectric–magnetic materials began to record an impressive rate of growth from mid–2000, after the announcement of an experimental result that could not be explained in any other way [1]. The underlying reason for negative refraction is that the time–averaged Poynting vector and the wavevector of a plane wave are oppositely aligned in those materials, for which reason we call them *negative–phase–velocity* (NPV) materials [2]. The promise of industrial exploitation of negative refraction has midwifed the extension of theoretical studies to bianisotropic materials [2–4]. The hope is that nanotechnological routes will allow negative refraction in the optical regime, which would be a truly remarkable accomplishment.

The potential consequences of NPV propagation in astrophysical settings are far–reaching, ranging from fundamental cosmological issues to space communication technologies [5]. The classical electromagnetic vacuum cannot support NPV plane waves, as it appears the same to all observers moving at constant relative velocities [6]. However, gravitational fields due to nearby massive objects would certainly distort electromagnetic propagation, which points towards the possibility of gravitationally assisted NPV propagation in vacuum. Investigation of a special case showed that this possibility cannot be discounted in spacetime manifolds of limited extent wherein the metric can be assumed to be approximately uniform [7].

Our objective in this communication is to examine the facilitation of NPV propagation of electromagnetic waves by the cosmological constant. Applying the formalism presented elsewhere [7,5], we consider both de Sitter and anti–de Sitter spacetimes [8,9].

2 Cosmological spacetime

In a static (i.e., time-independent) cosmological spacetime, the matrix representation of the metric $g_{\alpha\beta}$ is expressed in Cartesian coordinates as[1] [8]

$$[\,g_{\alpha\beta}\,] = \begin{pmatrix} m & 0 & 0 & 0 \\ 0 & -\left(1+\dfrac{\Lambda x^2}{3c^2m}\right) & -\dfrac{\Lambda xy}{3c^2m} & -\dfrac{\Lambda xz}{3c^2m} \\ 0 & -\dfrac{\Lambda xy}{3c^2m} & -\left(1+\dfrac{\Lambda y^2}{3c^2m}\right) & -\dfrac{\Lambda yz}{3c^2m} \\ 0 & -\dfrac{\Lambda xz}{3c^2m} & -\dfrac{\Lambda yz}{3c^2m} & -\left(1+\dfrac{\Lambda z^2}{3c^2m}\right) \end{pmatrix}$$

[a] *Also affiliated with:* Department of Physics, Imperial College, London SW7 2BZ, UK

[1] Roman indexes take the values 1, 2 and 3; while Greek indexes take the values 0, 1, 2, and 3.

(1)

where

$$m = 1 - \frac{\Lambda\left(x^2+y^2+z^2\right)}{3c^2}\,, \tag{2}$$

and c is the speed of light in vacuum in the absence of a gravitational field. If the cosmological constant Λ is positive (negative), the spacetime is called de Sitter (anti-de Sitter) spacetime [9]. The apparent singularity at $m=0$ arises because of an infelicitous choice of coordinate system. The metric can actually be extended to a geodesically complete space of constant curvature in other coordinate systems [10].

The electromagnetic response of vacuum in curved spacetime may be described by the constitutive relations of an equivalent, instantaneously responding, medium as per [7,11–13]

$$\left.\begin{array}{l} \underline{D} = \epsilon_0 \underline{\underline{\gamma}} \bullet \underline{E} \\ \underline{B} = \mu_0 \underline{\underline{\gamma}} \bullet \underline{H} \end{array}\right\}, \tag{3}$$

wherein SI units are implemented. Here, $\epsilon_0 = 8.854\times 10^{-12}$ F m $^{-1}$, $\mu_0 = 4\pi\times 10^{-12}$ H m^{-1}, and $\underline{\underline{\gamma}}$ is the 3×3 dyadic equivalent of the metric $[\,\gamma_{ab}\,]$ with components

$$\gamma_{ab} = -\frac{g^{ab}}{g_{00}}. \tag{4}$$

The constitutive relations (3) provide a global description of cosmological spacetime. Let us partition the global spacetime into neighbourhoods [5,14]. We focus our attention upon a neighbourhood $\mathcal{R}$ of the arbitrary location $(\tilde{x},\tilde{y},\tilde{z})$ wherein the nonuniform metric γ_{ab} may be approximated by the uniform metric $\tilde{\gamma}_{ab}$. Thus, we have the uniform 3×3 dyadic representation

$$\underline{\underline{\tilde{\gamma}}} \equiv [\,\tilde{\gamma}_{ab}\,] = \frac{1}{\tilde{m}} \begin{pmatrix} 1-\frac{\Lambda\tilde{x}^2}{3c^2} & -\frac{\Lambda\tilde{x}\tilde{y}}{3c^2} & -\frac{\Lambda\tilde{x}\tilde{z}}{3c^2} \\ -\frac{\Lambda\tilde{x}\tilde{y}}{3c^2} & 1-\frac{\Lambda\tilde{y}^2}{3c^2} & -\frac{\Lambda\tilde{y}\tilde{z}}{3c^2} \\ -\frac{\Lambda\tilde{x}\tilde{z}}{3c^2} & -\frac{\Lambda\tilde{y}\tilde{z}}{3c^2} & 1-\frac{\Lambda\tilde{z}^2}{3c^2} \end{pmatrix} \tag{5}$$

in $\mathcal{R}$, where the constant

$$\tilde{m} = 1 - \frac{\Lambda\left(\tilde{x}^2+\tilde{y}^2+\tilde{z}^2\right)}{3c^2}\,. \tag{6}$$

3 Plane waves in $\mathcal{R}$

Planewave solutions

$$\underline{E} = \mathrm{Re}\left\{\underline{E}_{\,0} \exp\left[i\left(\underline{k}\bullet\underline{r}-\omega t\right)\right]\right\}\,, \tag{7}$$
$$\underline{H} = \mathrm{Re}\left\{\underline{H}_{\,0} \exp\left[i\left(\underline{k}\bullet\underline{r}-\omega t\right)\right]\right\}\,, \tag{8}$$

are sought to the source–free Maxwell curl postulates

$$\nabla\times\underline{E} + \frac{\partial}{\partial t}\underline{B} = \underline{0}\,, \tag{9}$$

$$\nabla\times\underline{H} - \frac{\partial}{\partial t}\underline{D} = \underline{0} \tag{10}$$

in $\mathcal{R}$. Here, $\underline{k}$ is the wavevector, $\underline{r}$ is the position vector within the neighbourhood containing $(\tilde{x},\tilde{y},\tilde{z})$, ω is the angular frequency, and t denotes the time; whereas $i=\sqrt{-1}$, and $\underline{E}_{\,0}$ as well as $\underline{H}_{\,0}$ are complex–valued amplitudes. By combining (7)–(10) we find, after some manipulation, that

$$\underline{\underline{W}} \bullet \underline{E}_{\,0} = \underline{0}\,, \tag{11}$$

where

$$\underline{\underline{W}} = \left(k_0^2 \det\left[\underline{\underline{\tilde{\gamma}}}\right] - \underline{k}\bullet\underline{\underline{\tilde{\gamma}}}\bullet\underline{k}\right)\underline{\underline{I}} + \underline{k}\,\underline{k}\bullet\underline{\underline{\tilde{\gamma}}}, \tag{12}$$

and $k_0 = \omega\sqrt{\epsilon_0\mu_0}$. The corresponding dispersion relation $\det\left[\,\underline{\underline{W}}\,\right]=0$ is expressible as

$$k_0^2 \det\left[\underline{\underline{\tilde{\gamma}}}\right]\left(k_0^2\det\left[\underline{\underline{\tilde{\gamma}}}\right] - \underline{k}\bullet\underline{\underline{\tilde{\gamma}}}\bullet\underline{k}\right)^2 = 0\,. \tag{13}$$

Thus, the wavevectors satisfy the condition

$$\underline{k}\bullet\underline{\underline{\tilde{\gamma}}}\bullet\underline{k} = k_0^2\det\left[\underline{\underline{\tilde{\gamma}}}\right]\,, \tag{14}$$

as long as $\underline{\underline{\tilde{\gamma}}}$ is nonsingular.

Now we turn to the eigensolutions of (11). By virtue of (14), we have

$$\underline{k}\,\underline{k}\bullet\underline{\underline{\tilde{\gamma}}}\bullet\underline{E}_{\,0} = \underline{0}\,; \tag{15}$$

hence, $\underline{E}_{\,0}$ is orthogonal to $\underline{k}\bullet\underline{\underline{\tilde{\gamma}}}$. Without any loss of generality, let us choose the wavevector $\underline{k}$ to lie along the z axis, i.e.,

$$\underline{k} = k\underline{\hat{u}}_z\,, \tag{16}$$

with the unit vector $\underline{\hat{u}}_z$ lying along the Cartesian z axis. Thereby,

$$\underline{k}\bullet\underline{\underline{\tilde{\gamma}}} = k(\tilde{\gamma}_1\underline{\hat{u}}_x + \tilde{\gamma}_2\underline{\hat{u}}_y + \tilde{\gamma}_3\underline{\hat{u}}_z)\,, \tag{17}$$

with $\underline{\hat{u}}_x$ and $\underline{\hat{u}}_y$ being unit vectors lying along the Cartesian x and y axes, respectively, and

$$\tilde{\gamma}_1 = -\frac{\Lambda\tilde{x}\tilde{z}}{3c^2\tilde{m}}\,,\quad \tilde{\gamma}_2 = -\frac{\Lambda\tilde{y}\tilde{z}}{3c^2\tilde{m}}\,,\quad \tilde{\gamma}_3 = \frac{\tilde{m}_z}{\tilde{m}}\,, \tag{18}$$

wherein

$$\tilde{m}_z = 1 - \frac{\Lambda\tilde{z}^2}{3c^2}\,. \tag{19}$$

Two linearly independent eigenvectors satisfying (15) are provided as

$$\underline{e}_{\,1} = \tilde{\gamma}_2\underline{\hat{u}}_x - \tilde{\gamma}_1\underline{\hat{u}}_y\,, \tag{20}$$
$$\underline{e}_{\,2} = \tilde{\gamma}_1\tilde{\gamma}_3\underline{\hat{u}}_x + \tilde{\gamma}_2\tilde{\gamma}_3\underline{\hat{u}}_y - \left(\tilde{\gamma}_1^2+\tilde{\gamma}_2^2\right)\underline{\hat{u}}_z\,; \tag{21}$$

hence, the general solution is given by

$$\underline{E}_{\,0} = C_1\underline{e}_{\,1} + C_2\underline{e}_{\,2}\,, \tag{22}$$

wherein C_1 and C_2 are arbitrary complex–valued constants. The corresponding expression for $\underline{H}_0$ follows from the Maxwell postulates as

$$\underline{H}_0 = \frac{k}{\omega\mu_0} \left(C_1 \tilde{m} \underline{e}_2 - C_2 \tilde{m}_z \underline{e}_1 \right) . \tag{23}$$

The wavenumbers emerge straightforwardly from the dispersion equation (14). For $\underline{k}$ aligned with the z axis we obtain the k–quadratic expression

$$k^2 \frac{\tilde{m}_z}{\tilde{m}} - \frac{k_0^2}{\tilde{m}^2} = 0 , \tag{24}$$

which yields the wavenumbers

$$k = \pm k_0 \left(\tilde{m}\tilde{m}_z \right)^{-1/2} . \tag{25}$$

The requirement that $k \in \mathbb{R}$ imposes the condition

$$\tilde{m}\tilde{m}_z > 0 ; \tag{26}$$

in other words, both $\tilde{m}$ and $\tilde{m}_z$ must be of the same signs for real–valued k.

4 NPV Condition

In order to establish the feasibility of NPV propagation, we turn to the time–averaged Poynting vector given by

$$\langle \underline{P} \rangle = \frac{1}{2} \mathrm{Re} \left\{ \underline{E}_0 \times \underline{H}_0^* \right\} . \tag{27}$$

The general solution (22) and (23) delivers

$$\langle \underline{P} \rangle = \frac{k}{2\,\omega\mu_0} \left(|C_1|^2 \tilde{m} + |C_2|^2 \tilde{m}_z \right) \underline{e}_1 \times \underline{e}_2 . \tag{28}$$

From (20) and (21) we have

$$\underline{e}_1 \times \underline{e}_2 = \left(\gamma_1^2 + \gamma_2^2 \right) \left(\gamma_1 \underline{\hat{u}}_x + \gamma_2 \underline{\hat{u}}_y + \gamma_3 \underline{\hat{u}}_z \right) ; \tag{29}$$

thus

$$\underline{k} \bullet \left(\underline{e}_1 \times \underline{e}_2 \right) = k \frac{\tilde{m}_z}{\tilde{m}} \left(\frac{\Lambda \tilde{z}}{3c^2 \tilde{m}} \right)^2 \left(\tilde{x}^2 + \tilde{y}^2 \right) \tag{30}$$

and

$$\underline{k} \bullet \langle \underline{P} \rangle = \frac{1}{2\,\omega\mu_0} \left(\frac{k \Lambda \tilde{z}}{3c^2 \tilde{m}} \right)^2 \left(\tilde{x}^2 + \tilde{y}^2 \right) \times \left(|C_1|^2 + |C_2|^2 \frac{\tilde{m}_z}{\tilde{m}} \right) \tilde{m}_z . \tag{31}$$

The definition of NPV is that $\underline{k} \bullet \langle \underline{P} \rangle < 0$ [6,7]. By exploiting (26), we see that $\underline{k} \bullet \underline{P} < 0$ follows from $\tilde{m}_z < 0$; i.e., NPV propagation arises provided that

$$\Lambda > \frac{3c^2}{\tilde{z}^2} . \tag{32}$$

According to the NPV condition (32), the anti–de Sitter spacetime (i.e., $\Lambda < 0$) does not support NPV propagation, unlike the de Sitter spacetime (i.e., $\Lambda > 0$) which supports NPV propagation as long as the cosmological constant is sufficiently large. Equivalently, NPV propagation is feasible in de Sitter spacetime at sufficiently remote locations. We observe that de Sitter NPV propagation is supported at locations which lie outside the event horizon specified by $\left(x^2 + y^2 + z^2 \right) = 3c^2/\Lambda$.

5 Applicability of NPV condition

Let us comment now on the applicability of the NPV condition (32). Suppose δ is some representative linear dimension of the neighbourhood $\mathcal{R}$. The uniform approximation, as implemented for our planewave analysis, rests upon the assumptions that δ is (i) small compared with the radius of curvature of the global spacetime; and (ii) large compared with electromagnetic wavelength as given by $2\pi/k$. To be specific, consider the Ricci scalar R. For de Sitter spacetime we have [15]

$$R = \frac{4\Lambda}{c^2} . \tag{33}$$

As R provides a measure of the inverse radius of spacetime curvature squared, the neighbourhood $\mathcal{R}$ is such that

$$\frac{2\pi}{|k|} \ll \delta \ll \frac{c}{2} \sqrt{\frac{\rho}{|\Lambda|}} , \tag{34}$$

where ρ is a proportionality constant. Thus, we see that the partition of global spacetime requires

$$|\Lambda| \ll \frac{c^2 |k|^2 \rho}{16\pi^2} . \tag{35}$$

Since the linear dimensions δ of the neighbourhood $\mathcal{R}$ are chosen independently of the $\tilde{z}$ coordinate specifying the location of $\mathcal{R}$, there is no incompatibility between the conditions imposed on Λ by the NPV inequality (32) and the uniform approximation inequalities (34).

After solving for the planewave propagation modes in each neighbourhood, the neighbourhood solutions may be stitched together to provide the global solution. The piecewise uniform approximation technique adopted here — which is described in detail elsewhere [14] — is commonly employed in solving differential equations with nonhomogeneous coefficients [16].

6 Discussion

We have theoretically examined the propagation of electromagnetic plane waves with the time–averaged Poynting vector directed in opposition to the wavevector in cosmological spacetime with piecewise uniform metric. Our derivations lead us to conclude that NPV propagation is possible for Λ sufficiently large and positive (de Sitter spacetime); but it is impossible for negative Λ, i.e., when the spacetime is of the anti–de Sitter type. This difference between the two types of spacetimes should be added to the catalogue of known differences for scalar waves [17].

The cosmological constant problem has long cast a shadow over theoretical attempts to describe to the universe [18]. Recent string–inspired theories with *large* extra dimensions, known as brane–world models [19–21], suggest that gravitational interactions between particles on the brane in uncompactified bulk five-dimensional space can exhibit the correct four–dimensional brane behaviour, through finely tuning a relation between the bulk cosmological constant and the brane tension. In the original Randall–Sundrum model [20], the cosmological constant and the brane tension exactly cancel each other. More recent models, which admit the possibility of the cosmological constant and the brane tension not being in complete cancellation, yield a net cosmological constant on the four–dimensional brane that could be either positive or negative [22]. As a consequence of our results, one could envision an electromagnetic/optical experiment, based upon the negative refraction associated with NPV propagation [3,23], which may reveal whether or not a four–dimensional universe — deriving from a five–dimensional brane universe — has a sufficiently large positive cosmological constant.

The implications of NPV propagation in curved spacetime extend beyond the determination of the cosmological constant. For example, NPV propagation may facilitate the search for dark matter in the universe [24,25] and may shed light on other poorly understood phenomenons such as the anomalous acceleration of Pioneer 10 [24,26]. Also, possible applications of NPV propagation in space communication technologies may be envisaged.

Acknowledgements. SS acknowledges EPSRC for support under grant GR/S60631/01.

References

1. R.A. Shelby, D.R. Smith, S. Schultz, Science **292**, 77 (2001)
2. A. Lakhtakia, M.W. McCall, W.S. Weiglhofer, in: Introduction to Complex Mediums for Optics and Electromagnetics, edited by W.S. Weiglhofer and A. Lakhtakia (SPIE Press, Bellingham, WA, USA, 2003) pp 347–363
3. J.B. Pendry, Contemporary Phys. **45**, 191 (2004)
4. T.G. Mackay, A. Lakhtakia, Phys. Rev. E **69**, 026602 (2004)
5. T.G. Mackay, A. Lakhtakia, S. Setiawan, New J. Phys. **7**, 75 (2005)
6. T.G. Mackay, A. Lakhtakia, J. Phys. A: Math. Gen. **37**, 5697 (2004)
7. A. Lakhtakia, T.G. Mackay, J. Phys. A: Math. Gen. **37**, L505 (2004); corrections: **37**, 12093 (2004)
8. G.W. Gibbons, S.W. Hawking, Phys. Rev. D **15**, 2738 (1977)
9. H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, E. Herlt, Exact Solutions of Einstein's Field Equations, 2nd ed (Cambridge University Press, Cambridge, UK, 2003) Chap 8
10. S.W. Hawking, G.F.R. Ellis, The Large Scale Structure of Space-time (Cambridge University Press, Cambridge, UK, 1973) Chap 5
11. G.V. Skrotskii, Soviet Phys.–Dokl. **2**, 226 (1957)
12. J. Plebanski, Phys. Rev. **118**, 1396 (1960)
13. W. Schleich, M.O. Scully, in: New Trends in Atomic Physics, edited by G. Grynberg and R. Stora (Elsevier Science Publishers, Amsterdam, Holland, 1984) pp 995–1124
14. A. Lakhtakia, T.G. Mackay, S. Setiawan, Phys. Lett. A **336**, 89 (2005)
15. P.J.E. Peebles, Principles of Physical Cosmology (Princeton University Press, Princeton, NJ, USA, 1993)
16. J.D. Hoffman, Numerical Methods for Engineers and Scientists (McGraw–Hill, New York, USA, 1992)
17. Y.S. Myung, N.J. Kim, Class. Quantum Grav. **21**, 63 (2004)
18. M.R. Setare, Eur. Phys. J. C **38**, 373 (2004)
19. L. Randall, R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999)
20. L. Randall, R. Sundrum, Phys. Rev. Lett. **83**, 4690 (1999)
21. N. Arkani–Hamed, S. Dimopoulos, G. Dvali, Phys. Lett. B **429**, 263 (1998)
22. B. Bajc, G. Gabadadze, Phys. Lett. B **474**, 282 (2000)
23. J.B. Pendry, S.A. Ramakrishna, J. Phys.: Condens. Matter **15**, 6345 (2003)
24. Article entitled "13 things that don't make sense", New Scientist, March 19–25 (2005) pp 30–37
25. M. Sapinski, Eur. Phys. J. C **33**, s975 (2004)
26. J.D. Anderson, P.A. Laing, E.L. Lau, A.S. Liu, M.M. Nieto, S.G. Turyshev, Phys. Rev. D **65**, 082004 (2002)